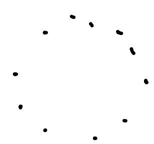
Mapper (inspired by Reeb graph)

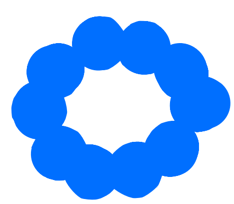
Motivating example:

Finite subset $P \subseteq \mathbb{R}^2$:



"like a circle"

Topologically: $P \not\cong S^1$



Replaced P by
$$P_r := \bigcup_{x \in P} B_r(x)$$

$P_r \cong S^1$

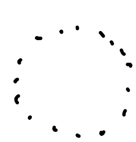
Problem: What is the "correct" $r > 0$?

If r is too small,
$P \simeq P_r$

If r is too large
$P_r \simeq *$

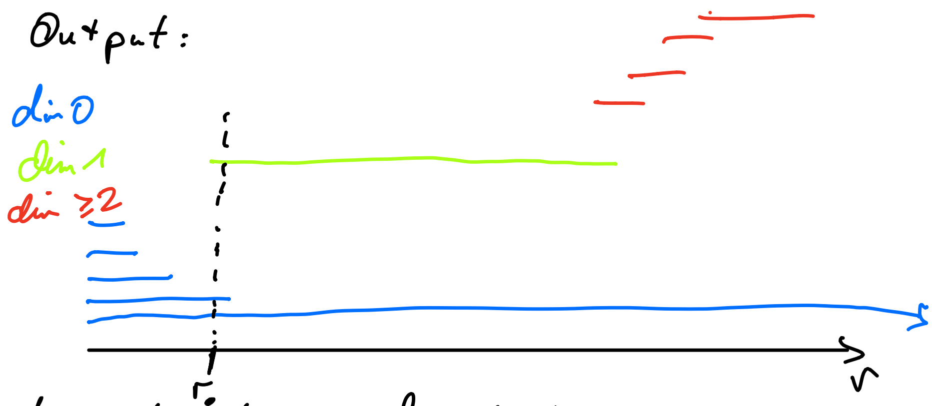
"Solution": We look at all r at once.

Input:



Output:

$H_*(S^1) = \begin{cases} \mathbb{k} & *=0,1 \\ 0 & \text{else} \end{cases}$

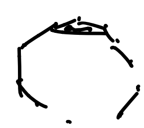


A collection of intervals ($\subseteq \mathbb{R}_{>0}$)

at r_0: 2 blue bars,
1 green bar

$\rightsquigarrow$ at r_0, we have 0-th Betti number
$= 2$
first Betti number
$= 1$

TDA-pipeline:



(1)

$P \longmapsto (K_r(P))_{r>0}$ with $K_{r_1}(P) \subseteq K_{r_2}(P)$ if $r_1 \leq r_2$

Rips complexes / Čech complexes

(2)

$(K_r)_{r>0} \longmapsto (H(K_r))_{r>0}$ with $H(K_{r_1}) \to H(K_{r_2})$ if $r_1 \leq r_2$

homology with field coeff.

③ $\mathbb{F}$

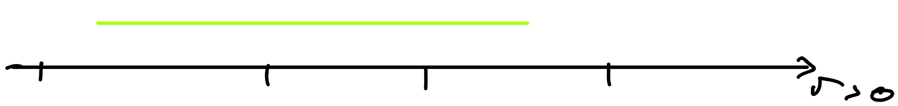
$$(H_r)_{r>0} \longmapsto \overbrace{(I_\alpha)_{\alpha \in A}}^{= \mathcal{B}(H_\bullet)} \qquad I_\alpha \subseteq \mathbb{R}_{>0} \text{ interval}$$

s.th. $H_\bullet \cong \bigoplus_{\alpha \in A} C(I_\alpha)$

$$C(I_\alpha)_r := \begin{cases} \mathbb{F} & \text{if } r \in I_\alpha \\ 0 & \text{else} \end{cases}$$

$$C(I_\alpha)_{r_1} \to C(I_\alpha)_{r_2} = \begin{cases} \mathrm{id}_{\mathbb{F}} & \text{if } r_1, r_2 \in I_\alpha \\ 0 & \text{else.} \end{cases}$$

$$0 \xrightarrow{0} \mathbb{F} \xrightarrow{\mathrm{id}} \mathbb{F} \xrightarrow{0} 0$$



Very important: We can compute this!

<u>Thm</u> (Chazal et al.)

W.r.t. suitable metrics,

$$P \mapsto \mathcal{B}(H(K_\bullet(P)))$$

is Lipschitz.

Research directions:

- Apply this to actual data.
- Extend this to multiple parameters.

$$\Big((H_{r,s}) \mapsto \square \quad \square \Big)$$

- Can we compute richer invariants?
- What can we learn about P from its barcode?
- Can we use barcodes in other areas of math?