# Multidimensional Persistence

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Persistent homology captures the topology of a filtration – a one- parameter family of increasing spaces – in terms of a complete discrete invariant. This invariant is a multiset of intervals that denote the lifetimes of the topological entities within the filtration. In many applications of topology, it is necessary to study a multifiltration: A family of spaces parameterized along multiple geometric dimensions (see [CZ09, Abstract]). In this Article, we summarize and comment on the results of [CZ09]: That no similar complete discrete invariant exists for multidimensional persistence. Instead, [CZ09] proposes the rank invariant, a discrete invariant for the robust estimation of Betti numbers in a multifiltration, and proves its completeness in one dimension.

## 1. Prelimaries

Let $k$ be a field and $A_n := k[x_1, \ldots, x_n]$ the polynomial ring in $n$ variables with the usual $\mathbb{Z}^n$-grading (or simply $n$-grading). $k$ becomes a $\mathbb{Z}^n$-graded module via $k_0 := k$ and $k_v = 0$ for $v \in \mathbb{Z}^n \setminus \{0\}$. For $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$, we define $x^v := x_1^{v_1} \cdot \ldots \cdot x_n^{v_n}$.

**Definition 1.1** (Multiset)**.** Let $S \subseteq \mathbb{Z}^n$ and $\mu : S \to \mathbb{N}$ be a map. $(S, \mu) := \{(s, i) \mid s \in S,\ i \leq \mu(s)\} \subseteq S \times \mathbb{N}$ is called multiset.

**Definition 1.2** (Quasiorder)**.** For $S \subseteq \mathbb{Z}^n$, $(s_1, ..., s_n), (t_1, ..., t_n) \in S$, define $(s_1, ..., s_n) \preceq (t_1, ..., t_n)$ if and only if $s_i \leq t_i$ for all $i \in \{1, ..., n\}$. For $S \subseteq \mathbb{Z}^n$, $(s, l), (t, k) \in (S, \mu)$ define $(s, l) \preceq (t, k)$ if and only if $s \preceq t$ for all $i \in \{1, ..., n\}$. $\mathbf{Vec}_k$ denotes the category of $k$-vector spaces with $k$-linear maps as morphisms.

**Remark 1.3.** For $S \subseteq \mathbb{N}_0^n$, $\preceq$ defines a quasi-partial order on $(S, \mu)$.

**Definition 1.4** (Persistence module)**.** A *persistence module* is a functor $M : \mathcal{C} \to \mathbf{Vec}_k$ where $\mathcal{C}$ is a small category.

1. $M$ is called *pointwise finite-dimensional* if for all $v \in \mathcal{C}$ $\dim_k(M_v) < \infty$.
2. $M$ is called $n$-dimensional if $\mathcal{C} = (\mathbb{Z}^n, \preceq)$.

Denote $\mathbf{Pers}_n$ the category of $n$-dimensional pointwise finite dimensional persistence modules $M$ with $M_v = 0$ for all $v \in \mathbb{Z}^n_{\prec 0}$ and $M_v = M_\delta$ for all $v \preceq \delta$ for some $\delta \in \mathbb{N}_0^n$, $\mathbf{Grfin}(A_n)$ the category of finetly generated $\mathbb{Z}^n$-graded modules over $A_n$ that are pointwise finite dimensional $k$-vector spaces and $\mathbf{Grfin}(A_n)_{\succeq 0} \subseteq \mathbf{Grfin}(A_n)$ the subcategory of objects that are positvely ($\geq 0$) graded.

**Definition 1.5** ([CZ09], Def.2, Structure)**.** Given $M \in \mathbf{Pers}_n$, we define a $n$-graded module over $A_n$ by

$$\alpha(M) := \bigoplus_{v \in \mathbb{Z}^n} M_v$$

where the $k$-module structure is the direct sum structure and $M_u \to M_v$ is $x^{v-u}$ for $u \preceq v$.

**Theorem 1.6** ([CZ09], Thm.1, Correspondence)**.** *The correspondence $\alpha$ defines an euqivalence of categories between $\mathbf{Pers}_n$ and $\mathbf{Grfin}(A_n)_{\succeq 0}$.*

**Theorem 1.7** ([CZ09],Thm. 2, refined)**.** *Let $k = \mathbb{F}_p$ for some prime $p$ and let $M \in \mathbf{Grfin}(A_n)_{\preceq 0}$. Then there is a multifiltered finite simplicial complex $X$ such that $H_l(X, k) \cong M$ for all $l \geq 0$.*

# 2. One-Dimensional Persistence

Let $k$ be a field and $M \in \mathbf{Pers}_1$, e.g. for $i \in \mathbb{N}_0$ $(H^i(X_j, k))_{j \in \mathbb{N}_0}$ the $i$-th homology of a bounded filtration $X_0 \subseteq ... \subseteq X_j = X_{j+1} = X$. We obtain according to [ZC05]

$$\mathbf{Pers}_1 \ni M \longmapsto \alpha(M) \cong \bigoplus_{i=1}^{n} \Sigma^{\alpha_i} k[t] \bigoplus_{j=1}^{m} \Sigma^{\gamma_j} k[t] \Big/ (t^{n_j}) \longmapsto B(M) \in \mathbf{Bar}$$

where the left isomorphism of finitely genereated graded $k[t]$-Modules is given by the standard *Structure Theorem for Persistent Modules* ($\Sigma^\alpha$ denotes an $\alpha$-shift upwards in grading).

$$B(M) := \bigcup_{i=1}^{n} \{([\alpha_i, \infty), k_i) \mid 1 \leq k_i \leq \mu_1(\alpha_i)\} \cup \bigcup_{j=1}^{m} \{([\lambda_j, \lambda_j + n_j), l_j) \mid 1 \leq l_i \leq \mu_2((\lambda_j, n_j))\}$$

denotes the *persistent barcode* of $M$ and **Bar** the category of Barcodes. The above assignments yields a complete classification (i.e. $B(M) = B(N)$ if and only if $N \cong M$) for the one-dimnensional case. In the following we discuss the question if it is possible to obtain a complete classification for $M \in \mathbf{Pers}_n(k)$ with $n \geq 2$.

# 3. Multidimensional Persistence

## 3.1. Complete Classification

**Definition 3.1** ([CZ09, §4.2], Shift, refined)**.** Given a $n$-graded $A_n$-Module $M$ and $v \in \mathbb{Z}^n$, the *shifted* $n$-graded $A_n$-module $M(v)$ is defined by $M(v)_u = M_{u-v}$ for all $u \in \mathbb{Z}^n$.

**Definition 3.2** ([CZ09, §4.2], Type)**.** Any $n$-graded $k$-vector space can be expressed as

$$V \cong \mathcal{V}((S_V, \mu_V)) := \bigoplus_{(v,i) \in (S_V, \mu_V)} k(v)$$

for a suitable multiset $(S_V, \mu_V)$. $\xi(V) := (S_V, \mu_V)$ is called the *type* of $V$. Analoguously any free $n$-graded $A_n$-module $F$ is isomorphic to

$$F \cong \mathcal{F}((S_F, \mu_F)) := \bigoplus_{(v,i) \in (S_F, \mu_F)} A_n(v)$$

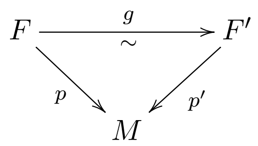
for a suitable finite multiset $(S_F, \mu_F)$. $\xi(F) := (S_F, \mu_F)$ is called the *type* of $F$.

**Definition 3.3** ([CZ09, §4.4, Def.4], Free hull)**.** For $M \in \mathbf{Grfin}(A_n)$, a *free hull* for $M$ is a surjective homomorphism $p : F \to M$ of $n$-graded modules, where $F \in \mathbf{Grfin}(A_n)$ is free, such that

$$id_k \otimes_{A_n} p : k \otimes_{A_n} F \to k \otimes_{A_n} M$$

is an isomorphism.

**Theorem 3.4** ([CZ09, §4.4, Thm.7]). *Every $M \in \mathbf{Grfin}(A_n)$ admits a free hull. Moreover, any two free hulls for $M$ are isomorphic in the sense that if $p : F \to M$ and $p' : F' \to M$ are both free hulls, there exists an isomrphism $g : F \to F'$ of $n$-graded modules s.t. the following diagram commutes:*




**Definition 3.5** ([CZ09, §4.5]). Let $M \in \mathbf{Grfin}(A_n)$ and $p : F \to M$ be a free Hull for $M$ with $K := \mathrm{Ker}(p)$. We define $\xi_0(M) := \xi(k \otimes_{A_n} M)$, $\xi_1(M) := \xi(k \otimes_{A_n} K)$.

**Theorem 3.6** ([CZ09, §4.5, Thm.8]). *For $M \in \mathbf{Grfin}(A_n)$, $\xi_0(M)$ and $\xi_1(M)$ are independent of the chosen free hull, hence they are a multiset-valued invariant of the isomorphism class of $M$.*

*Proof.* For $\xi_0(M)$ the assertion follows from Theorem 3.4. Suppose that $p : F \to M$ and $p' : F' \to M$ are free hulls for $M$. By Theorem 3.4 there is an isomorphism $\phi : M \to M'$ s.t. $p' \circ \phi = p$ which implies $\ker(p) \cong \ker(p')$ via restricting and we get $\xi(k \otimes_{A_n} \ker(p)) = \xi(k \otimes_{A_n} \ker(p'))$. ☐

**Remark 3.7.** For $M \in \mathbf{Pers}_1$, $\xi_0(M)$ and $\xi_1(M)$ correspond to the barcode $\mathrm{B}(M)$.

**Remark 3.8.** The Problem is that $\xi_0(M)$ and $\xi_1(M)$ are not complete for $M \in \mathbf{Grfin}$. Consider $M_1 = {}^{N_1}/_{T_1}$ and $M_2 = {}^{N_2}/_{T_2}$ where

$$\begin{aligned} N_1 &= N_2 = A_2 \oplus A_2, \\ T_1 &= \left(x_1^3, x_1^2 x_2, x_1^2 x_2, x_1 x_2^2, x_2^3\right) \oplus 0 \subseteq N_1, \\ T_2 &= \left(x_1^3, x_1^2 x_2\right) \oplus \left(x_1 x_2^2, x_2^3\right) \subseteq N_2. \end{aligned}$$

Then we have

$$\begin{aligned} \xi_0(M_1) = \xi_0(M_2) &= \{((0,0),1),((0,0),2)\}, \\ \xi_1(M_1) = \xi_1(M_2) &= \{((3,0),1),((2,1),1),((1,2),1),((0,3),1)\}, \end{aligned}$$

but $M_1$ and $M_2$ are not isomorphic.

Denote $\mathcal{I}(\xi_0, \xi_1)$ the set of isomorphism classes $[M]$ of $M \in \mathbf{Grfin}(A_n)$ with $\xi_0 = \xi_0(M)$ and $\xi_1 = \xi_1(M)$. Denote $F = \mathcal{F}(\xi_0)$ the free finitely generated $n$-graded $A_n$-module over the multiset $\xi_0$ and $\mathcal{S}(\xi_0, \xi_1)$ the set of all $n$-graded $A_n$-submodules $L \subseteq F$ which satisfy $\xi(k \otimes_{A_n} L) = \xi_1$ . $\mathrm{Aut}(F)$ acts on $\mathcal{S}(\xi_0, \xi_1)$ via $g \cdot L = g(L)$ for $g \in \mathrm{Aut}(F)$. We define a map

$$q_{\xi_0,\xi_1} : \mathcal{S}(\xi_0, \xi_1) \longrightarrow \mathcal{I}(\xi_0, \xi_1),\ L \longmapsto \left[{}^{F}/_{L}\right].$$

**Remark 3.9.** We need a condition that makes $F$ the free hull of ${}^{F}/_{L}$ which is not true in general. Otherwise, it would not be clear if the map $F \longmapsto \left[{}^{F}/_{L}\right]$ is well-defined. Consider for instance $L = A_n$ as submodule of $F = A_n \oplus A_n$ via embedding into the first component. Then we have ${}^{F}/_{L} \cong k$. After tensoring we get $k \otimes_{A_n} F \cong k^2$ and $k \otimes_{A_n} {}^{F}/_{L} \cong k$ which shows that $F$ is not the free hull of ${}^{F}/_{L}$. A condition to fix the problem could be to assume that $\mathrm{id}_k \otimes_{A_n} i : k \otimes_{A_n} L \to k \otimes_{A_n} F$ is the zero map where $i : L \to F$ denotes the canonical inclusion.

**Theorem 3.10** ([CZ09, §4.5, Thm.9], Classification). *The map $q_{\xi_0,\xi_1}$ satisfies the formula $q_{\xi_0,\xi_1}(g \cdot L) = q_{\xi_0,\xi_1}(L)$ and consequently induces a map*

$$\overline{q_{\xi_0,\xi_1}} : \mathcal{S}(\xi_0, \xi_1) \big/ \mathrm{Aut}(F) \longrightarrow \mathcal{I}(\xi_0, \xi_1).$$

*where* $\mathcal{S}(\xi_0,\xi_1) \Big/ \mathrm{Aut}(F) := \{\mathrm{Aut}(F)\cdot L \mid L \in \mathcal{S}(\xi_0,\xi_1)\}$ *denotes the orbit space. Moreover,* $\overline{q_{\xi_0,\xi_1}}$ *is bijective.*

*Proof.*

1. For $g \in \mathrm{Aut}(F)$ we have a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\quad g \quad} & F \\ \downarrow & & \downarrow \\ F\big/L & \xrightarrow[\sim]{\quad \overline{g} \quad} & F\big/g(L) \end{array}$$

   where $\overline{g}$ is an isomorphism and thus $q_{\xi_0,\xi_1}(g\cdot L) = q_{\xi_0,\xi_1}(L)$.

2. For surjectivity it suffices to show that $q_{\xi_0,\xi_1}$ is surjective. Let $M \in \mathcal{I}(\xi_0,\xi_1)$. By Theorem 3.4 there exists a surjection $p : F \to M$. We have $\xi(k \otimes_{A_n} \ker(p)) = \xi_1(M) = \xi_1$ by assumption, thus $\ker(p) \in \mathcal{S}(\xi_0,\xi_1)$ and clearly $q_{\xi_0,\xi_1}(\ker(p)) \in \mathcal{I}(\xi_0,\xi_1)$ which shows surjectivity.

3. For injectivity, we suppose that we are given $L, L' \in \mathcal{S}(\xi_0,\xi_1)$ with $q_{\xi_0,\xi_1}(L) = q_{\xi_0,\xi_1}(L')$, i.e. it exists an isomorphism $\alpha : F\big/L \to F\big/L'$. By Theorem 3.4 $\alpha$ lifts to $\widetilde{\alpha} \in \mathrm{Aut}(F)$ s.t.

$$\begin{array}{ccc} F & \xrightarrow{\quad \widetilde{\alpha} \quad} & F \\ \downarrow & & \downarrow \\ F\big/L & \xrightarrow[\sim]{\quad \alpha \quad} & F\big/L' \end{array}$$

   commutes and therefore $L \cong_{\widetilde{\alpha}} L'$.

□

What we have seen in this section is that $M \in \mathbf{Grfin}(A_n)$ is completely classified by $\xi_0(M)$, $\xi_1(M)$ and $\mathcal{Q}(M) := \overline{q_{\xi_0(M),\xi_1(M)}}^{-1}(M)$, i.e. for $M, M' \in \mathbf{Grfin}(A_n)$ holds $M \cong_{\mathbf{Grfin}(A_n)} M'$ if and only if $\xi_0(M) = \xi_0(M')$, $\xi_1(M) = \xi_1(M')$ and $\mathcal{Q}(M) = \mathcal{Q}(M')$. The invariants $\xi_0(M)$ and $\xi_1(M)$ are discrete. Unfortunenately it turns out that $\mathcal{Q}(\cdot)$ yields a continouus and not a discrete invariant, which is shown in [CZ09, §5] by realizing $\mathcal{I}(\xi_0,\xi_1)$ as the orbit space of an algebraic group action.

## 3.2. Parametrization

In the following we summarize the results of [CZ09, §5]. The following definition is a refinement of the definition in [CZ09, §5.1] motivated by the extra condition which was necessary to make $q_{\xi_0,\xi_1}$ well defined, by Example 3.15 and the definition of relation families in ([CZ07, Def.9]) where [CZ07]) is an earlier and different version of [CZ09].

**Definition 3.11** ([CZ09, §5.1] , refined)**.** Let $\xi_0 = (V_0,\alpha_0)$, $\xi_1 = (V_1,\alpha_1)$ be finite multisets and $\delta : V_1 \to \mathbb{Z}_{\geq 0}$ a map. $\mathbf{ARR}_{\xi_1,\delta}(F(\xi_0))$ denotes the set of families $(L_v)_{v\in V_1}$ where $L_v \subseteq F(\xi_0)_v$ are $k$-linear such that for all $v \in V_1$:

1. $\dim_k(L_v) = \delta(v)$.
2. $\dim_k\left(L_v \Big/ \sum_{v' \prec v} x^{v-v'} L_{v'}\right) = \alpha_1(v)$
3. $v' \preceq v \Longrightarrow x^{v-v'} L_{v'} \subseteq L_v$.

4. $\mathrm{id}_k \otimes_{A_n} i : k \otimes_{A_n} L_v \to k \otimes_{A_n} F$ is the zero map where $i : L_v \to F$ denotes the canonical inclusion.

It remains to show if condition 4. in Definition 3.12 is already contained in 1.-3. or not. In [CZ09, §5.1] they claim a bijection between $\mathcal{S}(\xi_0, \xi_1)$ and $\mathbf{ARR}_{\xi_1,\delta}(F(\xi_0))$ for certain $\delta$. Adjusting the notions of [CZ07, §8.3] to the situation in [CZ09] in order to get an analogous result I suggest the following:

**Proposition 3.12.** *For $\xi_0 = (V_0, \alpha_0)$ , $\xi_1 = (V_1, \alpha_1)$ finite mulitsets and $\delta = \dim_k \left(F(\xi_1)_{(\cdot)}\right)$, the following map is a bijection*

$$\begin{aligned} \mathcal{S}(\xi_0, \xi_1) &\xrightarrow{\sim} \mathbf{ARR}_{\xi_1,\delta}(F(\xi_0)), \\ L &\longmapsto (L_v)_{v \in V_1} \\ \sum_{v \in V_1} (L_v)_{A_n} &\longleftarrow (L_v)_{v \in V_1} \end{aligned}$$

**Remark 3.13.** The map in the above propostion is not given explicitly in [CZ09].

For a vectorspace $W$ of dimension $n$ denote $\mathbf{Gr}_d(W) := \{U \subseteq W \mid \dim_k(U) = d\}$ the *Grassmannian.*

**Theorem 3.14** ([CZ09, §5.1], Parametrization)**.** *$\mathbf{ARR}_{\xi_1,\delta}(F(\xi_0))$ identifies with a subprevariety of the projective variety $\prod_{v \in V} \mathbf{Gr}_{\delta(v)}(F_v)$ and in particular for $\delta = \dim_k \left(F(\xi_1)_{(\cdot)}\right)$, $\mathcal{S}(\xi_0, \xi_1) \cong \mathbf{ARR}_{\xi_1,\delta}(F(\xi_0))$ becomes a prevariety in a natural way. Moreover, the group action*

$$\mathrm{Aut}(F) \curvearrowright \mathcal{S}(\xi_0, \xi_1)$$

*is an algebraic group action.*

*Proof.* The basic idea in [CZ09, §5.1] is to interprete the (containment) conditions 1.-3. in Definition 3.11 algebraically. But the question that arises from Remark 3.9 is how to translate the additional condition which is necessary to make the assignment $L \to F/L$ well defined into an algebraic condition (if the condition is not optional in Definition 3.12). By [Thm.5]Carlsson2009 one can equip $\mathrm{Aut}(F)$ with an algebraic group structure. ☐

**Example 3.15** ([CZ09, §5.2], Continouus invariant, refined)**.** Consider $M \in \mathbf{Grfin}(A_n)$ as in Remark 3.8 with

$$\begin{aligned} \xi_0(M) &= \{((0,0),1),((0,0),2)\} \\ \xi_1(M) &= \{((3,0),1),((2,1),1),((1,2),1),((0,3),1)\} \end{aligned}$$

We have $\mathrm{Aut}(\mathcal{F}(\xi_0)) = \mathrm{GL}_2(k)$ by [CZ09, Thm.5]. For each $(v,i) \in \xi_1$ we have $\mathcal{F}(\xi_0)_v \cong_k k^2$ and $\dim_k(\mathcal{F}(\xi_1)) = 1$. One can easily check that the conditions 1.-4. in Definition 3.11 are trivial and hence

$$\mathbf{ARR}_{\xi_1,\dim_k\left(F(\xi_1)_{(\cdot)}\right)}(F(\xi_0)) = \prod_{v \in V_1} \mathbf{Gr}_{\dim_k(\mathcal{F}(\xi_1)_v)}(\mathcal{F}(\xi_0)_v) = \mathbf{Gr}_1\left(k^2\right)^4 = \mathbb{P}_1(k)^4$$

and therefore

$$\mathbb{P}_1(k)^4 \Big/ \mathrm{GL}_2(k) \cong \mathcal{I}(\xi_0, \xi_1).$$

Let $\Omega := \{(l_1, l_2, l_2, l_4) \in \mathbb{P}_1(k)^4 \mid l_i \neq l_j \,\text{for}\, i \neq j\} \subseteq \mathbb{P}_1(k)^4$ denote the subspace of pairwise distinct lines. $\Omega$ is clearly $\mathrm{GL}_2(k)$ invariant and by [CZ09, §5.3] we get

$$\mathbb{P}_1(k)^4 \Big/ \mathrm{GL}_2(k) \supseteq \Omega \Big/ \mathrm{GL}_2(k) = \mathbb{P}_1(k) \setminus \{0, 1, \infty\} = k \setminus \{0, 1\}.$$

This shows the field dependency and if $k$ is uncountable the continuous character of $\mathcal{Q}$.

### 3.3. The rank invariant

**Definition 3.16** ([CZ09, §6, Def.5], $\mathbb{D}^n$)**.** Let $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ with $u \leq \infty$ for all $u \in \overline{\mathbb{N}}$. Let $\mathbb{D}^n \subseteq \mathbb{N}^n \times \overline{\mathbb{N}}^n$ be the subset above the diagonal, i.e. $\mathbb{D}^n = \{(u, v) \mid u \in \mathbb{N}^n,\ v \in \overline{N}^n,\ u \preceq v\}$. For $(u, v), (u', v') \in \mathbb{D}^n$, we define $(u, v) \preceq (u', v')$ if and only if $u \preceq u'$ and $v \preceq v'$.

**Remark 3.17.** $(\mathbb{D}^n, \preceq)$ is a quasi-partially ordered set.

**Definition 3.18** ([CZ09, §6, Def.6] Rank invariant $\rho_M$)**.** Let $M \in \mathbf{Grfin}(A_n)$. We define

$$\rho_M : \mathbb{D}^n \longrightarrow \mathbb{N},\ (u, v) \longmapsto \operatorname{rank}(x^{v-u} : M_u \to M_v)$$

**Remark 3.19.** The function $\rho_M$ is clearly a discrete invariant for $M \in \mathbf{Grfin}(A_n)$.

**Lemma 3.20** ([CZ09, §6, Lem.7] Order-preserving)**.** *For all $(u, v), (u', v') \in \mathbb{D}^n$ it holds that $(u, v) \preceq (u', v')$ implies $\rho_M(u, v) \leq \rho_M(u', v')$. Therefore, $\rho_M$ is an order-preserving function from $(\mathbb{D}^n, \preceq)$ to $(\mathbb{N}, \leq)$.*

*Proof.* We have $\operatorname{rank}(f \circ g) \leq \operatorname{rank}(f), \operatorname{rank}(g)$ for any linear transformations $f, g$. □

**Theorem 3.21** ([CZ09, §6, Thm.12])**.** *The rank invariant $\rho_M$ is complete for $M \in \mathbf{Grfin}(A_1)$.*

*Proof.* The idea is to show that

$$\Psi : \mathbf{Bar} \longrightarrow \mathbf{Rank},\ \Psi(\xi)(t, s) := \{\big((t', s'), i\big) \in \xi \mid (t, s) \subseteq (t', s')\}$$

is a bijection between the set of Barcodes **Bar** and the set of rank invariants **Rank**. □

## References


[CZ07] Gunnar Carlsson and Afra Zomorodian. The theory of multidimensional persistence. *Discrete and Computational Geometry*, 42:71–93, 06 2007.

[CZ09] Gunnar Carlsson and Afra Zomorodian. The theory of multidimensional persistence. *Discrete & Computational Geometry*, 42(1):71–93, Jul 2009.

[ZC05] Afra Zomorodian and Gunnar Carlsson. Computing persistent homology. *Discrete Comput. Geom.*, 33(2):249–274, 2005.