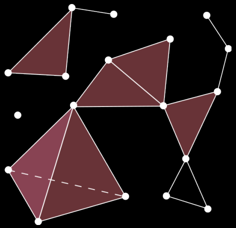

The Morse theory of Čech and Delaunay complexes

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This talk is mainly based on

Bauer & Edelsbrunner, *The Morse theory of Čech and Delaunay complexes* in Trans. Amer. Math. Soc. 369, 2017.

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Čech and Delaunay complexes

Let $X \subset \mathbb{R}^n$ be a point cloud. Čech complex:

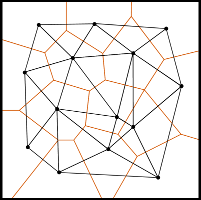
$$\check{\mathrm{C}}\mathrm{ech}_r(X) = \left\{ Q \subseteq X \,\middle|\, \bigcap_{x \in Q} B_r(x) \neq \emptyset \right\} \subseteq 2^X.$$

Voronoi balls:

$$\mathrm{Vor}_r(x, X) = B_r(x) \cap \{ y \in \mathbb{R}^n \mid d(y, x) \leq d(y, p) \text{ for all } p \in X \},$$

leading to the Delaunay complex

$$\mathrm{Del}_r(X) = \left\{ Q \subseteq X \,\middle|\, \bigcap_{x \in Q} \mathrm{Vor}_r(x, X) \neq \emptyset \right\}.$$



Voronoi diagram and its nerve, the Delaunay triangulation $\mathrm{Del}_\infty(X)$.

Delaunay-Čech complexes and relations

Delaunay-Čech complex:

$$\mathrm{Del\check{C}ech}_r(X) = \left\{ Q \in \mathrm{Del}_\infty(X) \,\middle|\, \bigcap_{x \in Q} B_r(x) \neq \emptyset \right\}.$$

Find that all complexes form a filtration, e.g., $\mathrm{Del}_r(X) \subseteq \mathrm{Del}_s(X)$ for all $r \leq s$. Clearly,

$$\mathrm{Del}_r(X) \subseteq \mathrm{Del\check{C}ech}_r(X) \subseteq \check{\mathrm{C}}\mathrm{ech}_r(X).$$

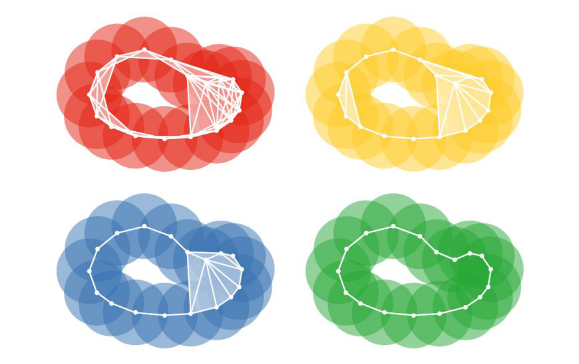
Wrap complex: subcomplex of Delaunay complex constructed from gradient of Delaunay radius function (not detailed here).

The Čech-Delaunay collapsing theorem

Theorem. Let X be a finite set of possibly weighted points in general position in $\mathbb{R}^n$. Then

$$\check{\mathrm{C}}\mathrm{ech}_r(X) \searrow \mathrm{Del}\check{\mathrm{C}}\mathrm{ech}_r(X) \searrow \mathrm{Del}_r(X) \searrow \mathrm{Wrap}_r(X)$$

for every $r \in \mathbb{R}$.



Simplicial complexes, reprinted from Bauer & Edelsbrunner 2017.

Embedding into literature

- generalizes and unifies previous Morse-theoretic treatments of selective Delaunay complexes

  e.g., Attali, Lieutier & Salinas in Comp. Geom. 46(4), 2013
- Continuous Morse theory for distance functions of finite point sets has been investigated

  e.g., Bobrowski & Adler in Homology, Homotopy and Applications 16(2), 2014

  $\Longrightarrow$ here, combinatorial structures from discrete gradient
- Extends collapse of Delaunay-Čech to Delaunay complex, which has been known before

  Bauer & Edelsbrunner in Journal of Computational Geometry 6(2), 2015.

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Discrete Morse theory I

For enjoyable intro to discrete Morse theory see Forman, *A user's guide to discrete Morse theory* in Séminaire Lotharingien de Combinatoire 48, 2002.

Recap. Finite set $X \subseteq \mathbb{R}^n$. Call subset $Q \subseteq X$ of $q+1$ points a q-simplex. Its faces are subsets of Q and facets are faces of dimension $q-1$. Simplicial complex is collection of simplices, K, closed under face relation.

Face relation defines canonical partial order on K.

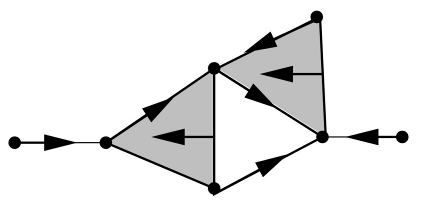
Hasse diagram $\mathcal{H}(K)$ is transitive reduction of this order, i.e., $\mathcal{H}(K)$ is the directed acyclic graph whose nodes are the simplices and arcs are pairs (P, Q) in which P is a facet of Q.

Discrete Morse theory II

Discrete vector field is a partition V of K into singleton sets $\{C\}$ and pairs $\{P, Q\}$ corresponding to arcs (P, Q) in the Hasse diagram.

Suppose function $f : K \to \mathbb{R}$ with $f(P) \leq f(Q)$ whenever P is face of Q with equality iff (P, Q) is pair in V. Then f is discrete Morse function and V its discrete gradient.

Simplex that does not belong to any pair in V is critical simplex and corresponding value of f is critical value of f.



Discrete gradient, reprinted from Forman 2002.

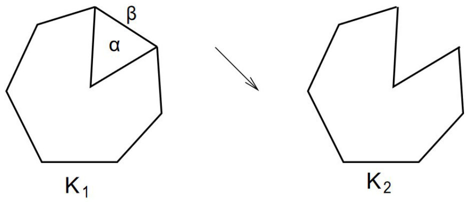
Discrete Morse theory III

Pairs in discrete gradient correspond to elementary collapses, realized continuously by a deformation retract. Thus, if can transform simpl. complex K to another K' using sequence of elementary collapses, then K and K' are homotopy-equivalent.

Here: stronger notion of simple-homotopy equivalence. Simplicial collapses instead of elementary ones.

Discrete Morse theory IV

Call a face $\tau \subseteq \sigma \subseteq K$ free, if σ is a facet of K and no other facet of K contains τ. A simplicial collapse of K is the removal of all simplices γ with $\tau \subseteq \gamma \subseteq \sigma$, where τ is a free face of σ.
If τ is facet of σ, then "elementary collapse".



A simplicial collapse.

Simplicial collapse, reprinted from Forman 2002.

Complex that has sequence of collapses leading to point is called collapsible. Every collapsible complex is contractible, but the converse is not true (cf. e.g. Bing's house or the dunce hat).

Outlook: Discrete Morse theory

Diverse fascinating applications of discrete Morse theory exist. Examples (following Forman 2002):

- The complex of not connected graphs: Vassiliev has shown how to derive finite type knot invariants from the study of the space of singular knots (i.e., maps from S^1 to $\mathbb{R}^3$ which are not embeddings) using discrete Morse theory.

  Vassiliev, arXiv.1409.5999

- Considerations of supersymmetry in quantum physics lead Witten to smooth Morse theory. Witten's derivation can be carried out in the discrete setting, too.

  Forman in Topology 37(5), 1998

- Investigations of certain Betti numbers of infinite simplicial complexes K which arise as a covering space of a finite simplicial complex K'.

  Mathai & Yates in Journal of functional analysis 168(1), 1999.

Generalized discrete Morse theory I

Interval in face relation of K is subset of form

$$[P, R] = \{Q \mid P \subseteq Q \subseteq R\}.$$

Call a partition W of K into intervals a generalized discrete vector field. Suppose function $f : K \to \mathbb{R}$ with $f(P) \leq f(Q)$ whenever P is face of Q, equality iff P and Q belong to common interval in W. Then, f is generalized discrete Morse function and W is its generalized discrete gradient.

Generalized discrete Morse theory II

If interval contains only one simplex, then "singular", the simplex is critical and the value of the simplex is a critical value of f.

For every generalized discrete gradient there is a discrete gradient refining every non-singular interval $[P, R]$ into pairs.

Theorem. Let K be a simplicial complex with a generalized discrete gradient V, and let $K' \subseteq K$ be a subcomplex. If $K \setminus K'$ is a union of non-singular intervals in V, then $K \searrow K'$.

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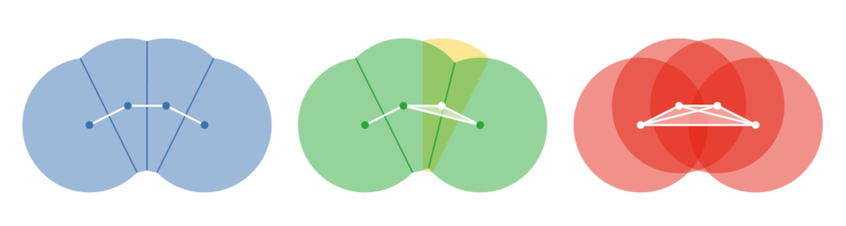
Selective Delaunay complexes I

Let $X \subset \mathbb{R}^n$ be finite set, $E \subseteq X$, $r \geq 0$. Define

$$\mathrm{Vor}_r(x, E) := B_r(x) \cap \{y \in \mathbb{R}^n \mid d(y, x) \leq d(y, p) \text{ for all } p \in E\}.$$

Selective Delaunay complex:

$$\mathrm{Del}_r(X, E) = \left\{ Q \subseteq X \;\middle|\; \bigcap_{x \in Q} \mathrm{Vor}_r(x, E) \neq \emptyset \right\}.$$



Selective Delaunay complexes, reprinted from Bauer & Edelsbrunner 2017.

Selective Delaunay complexes II

Note that

$$\mathrm{Del}_r(X, \emptyset) = \check{\mathrm{C}}\mathrm{ech}_r(X), \quad \mathrm{Del}_r(X, X) = \mathrm{Del}_r(X).$$

Define $\mathrm{Del}(X, E) := \mathrm{Del}_\infty(X, E)$.
Individual such Voronoi balls depend on E, but union does not.
Hence, nerve theorem implies that for given X and r, all selective Delaunay complexes have same homotopy type.

Radius functions

Consider $Q, E \subseteq X \subseteq \mathbb{R}^n$. A $(n-1)$-sphere $S \subset \mathbb{R}^n$ includes $Q \subseteq X$ if all points of Q lie on or inside S, it excludes $E \subseteq X$ if all points of E lie on or outside S. Set of such spheres may be empty, but if not, define Delaunay sphere $S(Q, E)$ with squared radius $s(Q, E)$ as smallest such sphere.

Radius function for E maps each simplex to squared radius of Delaunay sphere,

$$s_E : \mathrm{Del}(X, E) \to \mathbb{R}, \quad s_E(Q) = s(Q, E),$$

assuming $s(Q, E)$ exists.

Radius function lemma. Let $X \subseteq \mathbb{R}^n$ be finite, $E \subseteq X$ and $r \geq 0$. A simplex $Q \in \mathrm{Del}(X, E)$ belongs to $\mathrm{Del}_r(X, E)$ iff $s_E(Q) \leq r^2$.

Convex optimization I

Can determine whether or not a simplex Q belongs to $\mathrm{Del}_r(X,E)$ by solving a convex optimization problem:

$S(Q,E)$ is the sphere with center z and radius $r \geq 0$ that minimizes r^2 subject to the conditions

$$d(z,q)^2 \leq r^2 \,\forall q \in Q, \qquad d(z,e)^2 \geq r^2 \,\forall e \in E.$$

Generalize to weighted setting: associate weight $w_x \in \mathbb{R}$ to each point $x \in X$. Sphere S of (possibly negative) radius s with center z includes a point x with weight w_x if $d(z,x)^2 \leq s + w_x$, it excludes x if $d(z,x)^2 \geq s + w_x$. Then $S(Q,E)$ is the sphere that minimizes $s \in \mathbb{R}$ subject to the conditions

$$d(z,q)^2 \leq x + w_q \,\forall q \in Q, \qquad d(z,e)^2 \geq s + w_e \,\forall e \in E.$$

Convex optimization II: Karush-Kuhn-Tucker conditions

General optimization problem: minimize $f(y)$ subject to constraints

$$g_j(y) \leq 0 \, \forall j \in J, \qquad g_k(y) = 0 \, \forall k \in K \qquad g_l(y) \geq 0 \, \forall l \in L,$$

in which J, K, L are pairwise disjoint index sets.

Assuming f is convex and the g_i are affine, the **Karush-Kuhn-Tucker conditions** say that y is an optimal solution iff there exist coefficients $\lambda_i \in \mathbb{R}$ for all $i \in I = J \cup K \cup L$ such that

(i) stationarity: $\nabla f(y) + \sum_{i \in I} \lambda_i \nabla g_i(y) = 0$,

(ii) complementary slackness: $\lambda_i g_i(y) = 0$ for all $i \in I$,

(iii) dual feasibility: $\lambda_j \geq 0$ for all $j \in J$ and $\lambda_l \leq 0$ for all $l \in L$.

Convex optimization III: Recasting our problem

Introduce $a = ||z||^2 - s$, $y = (z, a)$. Set $K = Q \cap E$, $J = Q \setminus E$, $L = E \setminus Q$,

$$f(y) = s = ||z||^2 - a,$$

and the affine constraints for all $x \in Q \cup E$:

$$g_x(y) = ||z - x||^2 - s - w_x = -2\langle z, x\rangle + a + ||x||^2.$$

Defines optimization problem equivalent to the original one.

Convex optimization IV

Special KKT conditions. Let S be a sphere that includes $Q \subseteq X$ and excludes $E \subseteq X$. Then S is the smallest such sphere iff its center is an affine combination of the points $x \in Q \cup E$,

$$z = \sum \lambda_x x \quad \text{with} \quad 1 = \sum \lambda_x,$$

such that

(i) $\lambda_x = 0$ whenever x does not lie on S,

(ii) $\lambda_x \geq 0$ whenever $x \in Q \setminus E$, and

(iii) $\lambda_x \leq 0$ whenever $x \in E \setminus Q$.

Convex optimization V: combinatorial formulation

A circumsphere of a set $P \subseteq \mathbb{R}^n$ is an $(n-1)$-sphere such that all points of P lie on the sphere. If P is affinely independent, such a circumsphere exists. For sets of n of fewer points the circumsphere is not unique, but by the special KKT conditions a unique smallest circumsphere exists.

General position assumption. A finite set $X \subset \mathbb{R}^n$ is in general position if for every $P \subseteq X$ of at most $n+1$ points

(i) P is affinely independent,

(ii) no point of $X \setminus P$ lies on the smallest circumsphere of P.

Convex optimization VI: combinatorial formulation

X finite set of weighted points in general position. Let S be $(n-1)$-sphere. Write $\mathrm{Incl}\, S, \mathrm{Excl}\, S \subseteq X$ for the subsets of included and excluded points, $\mathrm{On}\, S = \mathrm{Incl}\, S \cap \mathrm{Excl}\, S$.

Assume S is smallest circumsphere of some set P, i.e., center z of S lies in affine hull of P and $P = \mathrm{On}\, S$ by general position. Have

$$z = \sum_{x \in \mathrm{On}\, S} \rho_x x \qquad \text{with} \qquad 1 = \sum_{x \in \mathrm{On}\, S} \rho_x .$$

By general position, affine combination is unique and $\rho_x \neq 0$ for all $x \in \mathrm{On}\, S$. Front face and back face of $\mathrm{On}\, S$:

$$\mathrm{Front}\, S = \{x \in \mathrm{On}\, S \mid \rho_x > 0\}, \qquad \mathrm{Back}\, S = \{x \in \mathrm{On}\, S \mid \rho_x < 0\}.$$

Have $\mathrm{Back}\, S = \emptyset$ iff circumcenter z is contained in convex hull of $\mathrm{On}\, S$.

Convex optimization VII: combinatorial KKT conditions

Theorem. Let $X \subset \mathbb{R}^n$ be a finite set of weighted points in general position. Let $Q, E \subseteq X$ for which there exists a sphere S with $Q \subseteq \mathrm{Incl}\, S$ and $E \subseteq \mathrm{Excl}\, S$. It is the smallest such sphere, $S = S(Q, E)$, iff

(i) S is the smallest circumsphere of $\mathrm{On}\, S$,

(ii) $\mathrm{Front}\, S \subseteq Q$,

(iii) $\mathrm{Back}\, S \subseteq E$.

Convex optimizatoin VIII: partition into intervals

Fix $E \subseteq X$; recall s_E maps $Q \in \mathrm{Del}(X, E)$ to squared radius of $S = S(Q, E)$. Implies $s_E(P) = s_E(Q)$ for all $P \in [\mathrm{Front}\, S, \mathrm{Incl}\, S]$.

To prove that s_E is generalized discrete Morse function remains to show that $s_E(P) < s_E(Q)$ whenever $P \subseteq Q$ do not belong to same interval. But this is clear from general position assumption.

Selective Delaunay Morse function theorem. Let $X \subset \mathbb{R}^n$ be finite set of weighted points in general position, $E \subseteq X$. Then $s_E : \mathrm{Del}(X, E) \to \mathbb{R}$ is generalized discrete Morse function whose discrete gradient consists of the intervals $[\mathrm{Front}\, S, \mathrm{Incl}\, S]$ over all Delaunay spheres $S = S(Q, E)$ with $Q \in \mathrm{Del}(X, E)$.

Convex optimization IX: partition into intervals

Call $Q \in \mathrm{Del}(X)$ a centered Delaunay simplex if the center of $S = S(Q, X)$ is contained in the convex hull of Q. In that case, $S = S(Q, E)$ for all sets $E \subseteq X$.

Critical simplex corollary. Let $X \subset \mathbb{R}^n$ be finite set of weighted points in general position. Independent of E, a subset $Q \subseteq X$ is a critical simplex of s_E iff $s(Q, \emptyset) = s(Q, E) = s(Q, X)$ iff Q is a centered Delaunay simplex.

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Strategy

Write $Q - x = Q \setminus \{x\}$ and $Q + x = Q \cup \{x\}$, one of these being equal to Q. Construct two discrete gradients. First one is defined on the full simplex of X and induces the simplicial collapse $\check{\mathrm{C}}\mathrm{ech}_r(X) \searrow \mathrm{Del}\check{\mathrm{C}}\mathrm{ech}_r(X)$ by removing all non-Delaunay simplices. The second discrete gradient is defined on $\mathrm{Del}(X)$ and induces the collapse $\mathrm{Del}\check{\mathrm{C}}\mathrm{ech}_r(X) \searrow \mathrm{Del}_r(X)$.

Gradients are constructed by assigning to each collapsed simplex $Q \in \mathrm{Del}_r(X, E) \searrow \mathrm{Del}_r(X, E)$ for $E \subseteq F \subseteq X$ a point $x \in F \setminus E$ that turns the sphere $S(Q, E)$ infeasible for the excluded set F. Thus, $S(Q, F)$ will either have a larger radius or not exist at all.

Pairing lemmas

First simplex pairing lemma. Let $E \subseteq F \subseteq X$ and $Q \in \mathrm{Del}(X, F)$ with $S(Q, E) \neq S(Q, F)$. Then there exists a point $x \in F \setminus E$ such that

(i) $S(Q - x, E) = S(Q + x, E)$,

(ii) $S(Q - x, F) = S(Q + x, F)$.

Second simplex pairing lemma. Let $E \subseteq F \subseteq X$ and let Q be a simplex in $\mathrm{Del}(X, E)$ but not in $\mathrm{Del}(X, F)$. Then there exists a point $x \in F \setminus E$ such that

(i) $S(Q - x, E) = S(Q + x, E)$,

(ii) both $Q - x$ and $Q + x$ are not in $\mathrm{Del}(X, F)$.

Collapsing I

Selective Delaunay collapsing theorem. Let $X \subset \mathbb{R}^n$ be a finite set of possibly weighted points in general position and let $E \subseteq F \subseteq X$. Then,

$$\mathrm{Del}_r(X, E) \searrow \mathrm{Del}_r(X, E) \cap \mathrm{Del}(X, F) \searrow \mathrm{Del}_r(X, F)$$

for every $r \in \mathbb{R}$.

Idea of proof: show that both collapses are induced by discrete gradients constructed with the help of the two simplex pairing lemmas and additional auxiliary lemmas (not presented here).

Collapsing II

Setting $E = \emptyset$, $F = X$, find from selective Delaunay collapsing theorem with additional arguments for the Wrap complex (not given here):

Čech-Delaunay collapsing theorem. Let $X \subset \mathbb{R}^n$ be a finite set of possibly weighted points in general position. Then

$$\mathrm{\check{C}ech}_r(X) \searrow \mathrm{Del\check{C}ech}_r(X) \searrow \mathrm{Del}_r(X) \searrow \mathrm{Wrap}_r(X)$$

for every $r \in \mathbb{R}$.

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Naturality and persistence I

A natural transformation from a filtration $(K_t)_{t\in\mathbb{R}}$ to another filtration $(L_t)_{t\in\mathbb{R}}$ is a family of continuous maps $K_t \to L_t$ such that the diagram

$$\begin{array}{ccc} K_r & \longrightarrow & L_r \\ \downarrow & & \downarrow \\ K_t & \longrightarrow & L_t \end{array}$$

commutes for all $r \leq t$. Persistent homology of $(K_t)_{t\in\mathbb{R}}$ is the diagram of homology groups $H_*(K_t)$ connected by the homomorphisms induced by inclusions $K_r \hookrightarrow K_t$ for $r \leq t$. Homology being a functor, it sends a natural transformation of filtrations to a natural transformation of their persistent homology.

Naturality and persistence II

By the Čech-Delaunay collapsing theorem, the diagram

$$\begin{array}{ccccccc}
\mathrm{Wrap}_r(X) & \hookrightarrow & \mathrm{Del}_r(X) & \hookrightarrow & \mathrm{Del\check{C}ech}_r(X) & \hookrightarrow & \check{\mathrm{C}}\mathrm{ech}_r(X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Wrap}_t(X) & \hookrightarrow & \mathrm{Del}_t(X) & \hookrightarrow & \mathrm{Del\check{C}ech}_t(X) & \hookrightarrow & \check{\mathrm{C}}\mathrm{ech}_t(X)
\end{array}$$

commutes for all $r \leq t$. Horizontal inclusion maps in this diagram correspond to collapses of the Čech-Delaunay collapsing theorem, i.e., inclusion maps constitute a natural transformation, which is a simple-homotopy equivalence at each filtration index. Thus,

Persistence isomorphism corollary. The Čech, Delaunay-Čech, Delaunay and Wrap filtrations have isomorphic persistent homology.